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# THE CONCENTRATION OF A WAVE FIELD AT THE INTERFACE OF ELASTIC MEDIA†

L. P. VOVK and B. V. SOBOL'

Rostov-on-Don

e-mail: bsobol@dstu.edu.ru; vovklp@ukrtop.com

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The steady symmetrical oscillations of a transversely non-uniform elastic rectangular region, consisting of three bonded uniform isotropic rectangles, are considered, where the elastic characteristics of the inner rectangle are assumed to be different from those of the external rectangles. Using methods developed previously [1, 2], the dependence of the order of the singularity of the stress field at the interface on the combinations of the constants of elasticity of the joined media and on their wave impedances is investigated. 

An analysis of the spectrum and form of the oscillations of an isotropic rectangle, based on the superposition method, was presented fairly completely in [3, 4]. A modification of this method was proposed in [1, 2], which improves its convergence as a result of investigating the features of the wave field at corner points. A quantitative investigation of the propagation of waves in a composite waveguide of infinite extent was carried out previously in [5–7].

## 1. FORMULATION OF THE PROBLEM

Suppose the cross-sections of a non-uniform elastic prism, infinite in the direction of the $\alpha_3$ axis, occupies the following region in the system of coordinates $\alpha_1 O \alpha_2$

$$D = G^{(1)} \cup G^{(2)} \tag{1.1}$$

The regions $G^{(m)}$ are joined to one another, isotropic, and in general have different constants of elasticity, and are defined by the inequalities

$$G^{(1)} = \{(\alpha_1, \alpha_2): |\alpha_1| \le c; |\alpha_2| \le b\},\ G^{(2)} = \{(\alpha_1, \alpha_2): \alpha_1 \in [-a, c] \cup [c, a]; |\alpha_2| \le b\} \tag{1.2}$$

Here and henceforth the superscript means that the mechanical characteristic or the modulus of elasticity belongs to the region $G^{(m)}$.

Suppose a load of intensity $q$, which varies harmonically with time with a frequency $\omega$, is specified on the external sides of the plate $\alpha_1 = \pm a$, $\alpha_2 = \pm b$. To investigate the frequencies and forms of the natural oscillations we will use the equations of motion, written in dimensionless functions and coordinates

$$\Delta U_\beta^{(m)} + (N_{12}^{(m)} + 1)(U_{1,1}^{(m)} + U_{2,2}^{(m)})_{,\beta} + \Omega^{(m)2} U_\beta^{(m)} = 0, \quad \beta = 1, 2;\ m = 1, 2 \tag{1.3}$$

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where

$$U_\beta^{(m)} = \frac{\upsilon_\beta^{(m)}}{a}, \quad N_{12}^{(m)} = N_{11}^{(m)} - 2, \quad N_{11}^{(m)} = \frac{2(1-\nu^{(m)})}{1-2\nu^{(m)}}, \quad f_{,1} = \frac{\partial f}{\partial x}, \quad f_{,2} = \frac{\partial f}{\partial y}$$
$$x = \frac{\alpha_1}{a}, \quad y = \frac{\alpha_2}{a}, \quad \Omega^{(m)2} = \frac{\rho^{(m)}\omega^2 a^2}{\mu^{(m)}} \tag{1.4}$$

$\mu^{(m)}$ is the shear modulus, $\nu^{(m)}$ is Poisson's ratio, $\rho^{(m)}$ is the density in the region $G^{(m)}$, and $\upsilon_\alpha^{(m)}$ are the amplitude components of the displacement vector. When formulating the boundary conditions we will take into account the symmetry of the region and we will consider the stressed state of the part of the region situated in the first quarter. We will introduce the following local dimensionless coordinate

$$\hat{x} = (\alpha_1 - c)/a, \quad \hat{x} \in [0, \delta_2]; \quad \delta_2 = 1 - \delta, \quad \delta = c/a \tag{1.5}$$

and the dimensionless amplitude components of the stress tensor $\sigma_{\alpha\beta}^{(m)}$

$$\sigma_{11}^{(m)} = N_{11}^{(m)} U_{1,1}^{(m)} + N_{12}^{(m)} U_{2,2}^{(m)}, \quad \sigma_{22}^{(m)} = N_{12}^{(m)} U_{1,1}^{(m)} + N_{11}^{(m)} U_{2,2}^{(m)}, \quad \sigma_{12}^{(m)} = U_{1,2}^{(m)} + U_{2,1}^{(m)} \tag{1.6}$$

Hence, the boundary conditions of the problem can be written in the following form:
in the region $\bar{G}^{(1)} = \{(x, y) : |x| \le \delta, \ |y| \le \eta\}$

$$\sigma_{1\beta}^{(1)}(\delta, y) = \sigma_{1\beta}^{(2)}(0, y), \quad U_\beta^{(1)}(\delta, y) = U_\beta^{(2)}(0, y), \quad \sigma_{22}^{(1)}(x, \eta) = q^{(1)}, \quad \sigma_{12}^{(1)}(x, \eta) = 0 \tag{1.7}$$

in the region $\bar{G}^{(2)} = \{(\hat{x}, y) : 0 \le \hat{x} \le \delta_2, \ |y| \le \eta\}$

$$\sigma_{11}^{(2)}(\delta_2, y) = q^{(2)}, \quad \sigma_{12}^{(2)}(\delta_2, y) = 0, \quad \sigma_{22}^{(2)}(\hat{x}, \eta) = q^{(2)}, \quad \sigma_{12}^{(2)}(\hat{x}, \eta) = 0 \tag{1.8}$$

Here

$$q^{(m)} = q/\mu^{(m)}, \quad \eta = b/a$$

## 2. THE SOLUTION OF AUXILIARY PROBLEMS

Using the superposition method [1–4] we will construct the general solution $U_\beta^{(m)}$, which satisfies the system of equations (1.3) inside the region $\bar{G}^{(m)}$, in the form of the sum of particular solutions of this system, each of which describes symmetrical oscillations of infinite strips, having a common region $\bar{G}^{(m)}$. Expressions for the particular solutions of the equations of motion, constructed for infinite layers, which represent sets of plane standing waves, will be fundamental for the investigation [3]. However, when choosing the form of these particular solutions it is necessary to take into account the possibility not only of satisfying the boundary conditions on the external boundary of the region when using them, but also the conditions for matching the mechanical characteristics at the interface of the media. This naturally requires changes in the numerical-analytical algorithm of the solution of the problem, proposed previously for solving the problem of forced oscillations of a uniform rectangle [2].

Thus, we will consider the solution of system (1.3) for the following auxiliary boundary conditions:
in the region $\bar{G}^{(1)}$

$$U_1^{(1)}(\delta, y) = f_1(y), \quad \sigma_{12}^{(1)}(\delta, y) = \varphi_1(y), \quad U_2^{(1)}(x, \eta) = f_2(x), \quad \sigma_{12}^{(1)}(x, \eta) = 0 \tag{2.1}$$

in the region $\bar{G}^{(2)}$

$$U_1^{(2)}(0, y) = f_1(y), \quad \sigma_{12}^{(2)}(0, y) = \varphi_1(y), \quad \sigma_{12}^{(2)}(\delta_2, y) = 0, \quad U_1^{(2)}(\delta_2, y) = f_3(y)$$
$$U_2^{(2)}(\hat{x}, \eta) = f_4(\hat{x}), \quad \sigma_{12}^{(2)}(\hat{x}, \eta) = 0 \tag{2.2}$$

Here $f_1(y), f_2(x), f_3(y), f_4(\hat{x}), \varphi_1(y)$ are unknown functions.

The choice of the auxiliary conditions in the form (2.1) and (2.2) is due to the fact that when formulating the auxiliary problem it is necessary not only to supplement its analytical solution, but also to introduce into the boundary conditions and the matching conditions the conditions from (1.7) and (1.8) as much as possible. This enables the form of the subsequent system of integral equations for determining the unknown functions introduced to be simplified as much as possible. Expanding them in Fourier series on the corresponding sections, after satisfying boundary conditions (2.1) and (2.2), we obtain, in the final analysis, after certain transformations of the solution of auxiliary problems (1.3), (2.1), (2.2) (everywhere henceforth the summation over $k$ and $j$ is carried out from unity to infinity)

$$\begin{aligned}
U_1^{(1)} &= \sum_k \frac{\alpha_k}{k_2^2}[2\alpha_k f_{1k}\Delta_1^{(1)}(x,\delta,\alpha_k) + \varphi_{1k}\Delta_4^{(1)}(x,\delta,\alpha_k)]\cos\alpha_k(y-\eta) + \\
&+ \sum_j \frac{2\beta_j^2}{k_2^2} f_{2j}\Delta_2^{(1)}(y,\eta,\beta_j)\sin\beta_j(x-\delta) + f_{10}\frac{\sin k_1}{\sin k_1\delta} \\
U_2^{(1)} &= \sum_k \frac{\alpha_k}{k_2^2}\left[2\alpha_k f_{1k}\Delta_2^{(1)}(x,\delta,\alpha_k) + \frac{\alpha_k\varphi_{1k}}{k_2^2}\Delta_3^{(1)}(x,\delta,\alpha_k)\right]\sin\alpha_k(y-\eta) + \\
&\qquad + \sum_j \frac{2\beta_j^2}{k_2^2} f_{2j}\Delta_1^{(1)}(y,\eta,\beta_j)\cos\beta_j(x-\delta) + f_{20}\frac{\sin k_1 y}{\sin k_1\eta} \\
U_1^{(2)} &= \sum_k \frac{\alpha_k}{l_2^2}[2\alpha_k f_{3k}\Delta_1^{(2)}(\hat{x},\delta_2,\alpha_k) - 2\alpha_k f_{1k}\Delta_1^{(2)}(\hat{x}-\delta_2,\delta_2,\alpha_k) - \\
&- \varphi_{1k}\Delta_4^{(2)}(\hat{x}-\delta_2,\delta_2,\alpha_k)]\cos\alpha_k(y-\eta) + \\
&+ \sum_j \frac{2\chi_j^2}{l_2^2} f_{4j}\Delta_2^{(2)}(y,\eta,\chi_j)\sin\chi_j(\hat{x}-\delta_2) + \frac{f_{30}\sin l_1\hat{x} - f_{10}\sin l_1(\hat{x}-\delta_2)}{\sin l_1\delta_2} \\
U_2^{(2)} &= \sum_k \frac{\alpha_k}{l_2^2}[2\alpha_k f_{3k}\Delta_2^{(2)}(\hat{x},\delta_2,\alpha_k) - 2\alpha_k f_{1k}\Delta_2^{(2)}(\hat{x}-\delta_2,\delta_2,\alpha_k) - \\
&- \varphi_{1k}\Delta_3^{(2)}(\hat{x}-\delta_2,\delta_2,\alpha_k)]\sin\alpha_k(y-\eta) + \sum_j \frac{2\chi_j^2}{l_2^2} f_{4j}\Delta_1^{(2)}(y,\eta,\chi_j)\cos\chi_j(\hat{x}-\delta_2) + f_{40}\frac{\sin l_1 y}{\sin l_1\eta}
\end{aligned} \tag{2.3}$$

Here

$$\alpha_k = k\pi/\eta, \quad \beta_k = k\pi/\delta, \quad \chi_k = k\pi/\delta_2$$

$$\begin{aligned}
&\Delta_1^{(m)}(x,y,z_j) = S_{2j}^{(m)} - \frac{a_{3j}^{(m)2}}{2z_j^2}S_{1j}^{(m)}, \quad \Delta_2^{(m)}(x,y,z_j) = \frac{a_{3j}^{(m)2}}{2z_j a_{1j}^{(m)}}C_{1j}^{(m)} - \frac{a_{2j}^{(m)}}{z_j}C_{2j}^{(m)} \\
&\Delta_3^{(m)}(x,y,z_j) = \frac{z_j}{a_{1j}^{(m)}}C_{1j}^{(m)} - \frac{a_{2j}^{(m)}}{z_j}C_{2j}^{(m)}, \quad \Delta_4^{(m)}(x,y,z_j) = S_{2j}^{(m)} - S_{1j}^{(m)} \\
&S_{\gamma j}^{(m)} = \frac{\operatorname{sh} a_{\gamma j}^{2} x}{\operatorname{sh} a_{\gamma j}^{(m)} y}, \quad C_{\gamma j}^{(m)} = \frac{\operatorname{ch} a_{\gamma j}^{(m)} x}{\operatorname{sh} a_{\gamma j}^{(m)} y}, \quad a_{\gamma j}^{(m)2} = z_j^2 - k_\gamma^{(m)2}, \quad a_{3j}^{(m)2} = a_{2j}^{(m)2} + z_j^2 \\
&k_1^{(m)} = \Omega^{(m)}/\sqrt{N_{11}^{(m)}}, \quad k_2^{(m)} = \Omega^{(m)}, \quad k_\gamma = k_\gamma^{(1)}, \quad l_\gamma = k_\gamma^{(2)}; \quad \gamma = 1,2; \quad m = 1,2
\end{aligned} \tag{2.4}$$

and $f_{10}, f_{1k}, \ldots, \varphi_{1k}$ are Fourier coefficients of the corresponding functions.

Using expressions (2.3), we can calculate the components of the stress tensor from formulae (1.6).

## 3. DERIVATION OF THE SYSTEM OF INTEGRAL EQUATIONS AND ITS ASYMPTOTIC ANALYSIS

Taking into account the unused boundary conditions from (2.1) and (2.2), namely,

$$\begin{aligned}&\sigma_{11}^{(1)}(\delta, y) = \sigma_{11}^{(2)}(0, y), \quad U_2^{(1)}(\delta, y) = U_2^{(2)}(0, y)\\&\sigma_{22}^{(1)}(x, \eta) = q^{(1)}, \quad \sigma_{11}^{(2)}(\delta_2, y) = q^{(2)}, \quad \sigma_{22}^{(2)}(\hat{x}, \eta) = q^{(2)}\end{aligned} \tag{3.1}$$

we will reduce the problem in question to the problem of solving the following system of integral equations in the functions $\varphi_1(y), f_1(y), f_2(x), f_3(y), f_4(\hat{x})$

$$M_{k1}\varphi_1 + \sum_{r=1}^{4} L_{kr} f_r = Q_k, \quad k = 1, 2, 3, 4, 5 \tag{3.2}$$

In system (3.2) the number of the equation is identical with the number of the boundary condition in formulae (3.1), and the corresponding operators are obtained from formulae (2.3), (2.4) and (1.6). For example

$$\begin{aligned}M_{11}\varphi_1 &= \sum_k \alpha_k \varphi_{1k} k_2^{-2}[\Delta_8^{(1)}(\delta, \delta, \alpha_k) + k_{12}^2 \Delta_8^{(2)}(-\delta_2, \delta_2, \alpha_k)]\cos\alpha_k(y-\eta)\\L_{11}f_1 &= \sum_k 2\alpha_k^2 f_{1k} k_2^{-2}[\Delta_7^{(1)}(\delta, \delta, \alpha_k) + k_{12}^2 \Delta_7^{(2)}(-\delta_2, \delta_2, \alpha_k)]\cos\alpha_k(y-\eta) + \\&+ (N_{11}^{(1)} k_1 \operatorname{ctg} k_1\delta + N_{11}^{(2)} l_1 \operatorname{ctg} l_1\delta_2) f_{10}\end{aligned} \tag{3.3}$$

Here

$$\begin{aligned}&k_{12} = \frac{k_2}{l_2} = \frac{\rho^{(1)}\mu^{(2)}}{\rho^{(2)}\mu^{(1)}}, \quad \Delta_7^{(m)}(x, y, z_k) = 2a_{2k}^{(m)} C_{2k}^{(m)} - \frac{a_{3k}^{(m)4}}{2a_{1k}^{(m)2} z_k} C_{1k}^{(m)}\\&\Delta_8^{(m)}(x, y, z_k) = 2a_{2k}^{(m)} C_{2k}^{(m)} - \frac{a_{3k}^{(m)2}}{a_{1k}^{(m)}} C_{1k}^{(m)}\end{aligned} \tag{3.4}$$

We will investigate the behaviour of the solution of system of integral equations (3.2) at the corner points of the regions $\bar{G}^{(m)}$. This enables us to determine the asymptotic form of the Fourier coefficients of the required functions $\varphi_{1k}, f_{1k}, f_{2j}, f_{3k}, f_{4j}$ for large values of the indices and successfully choose the coordinate functions [2] in Bubnov's method when solving system (3.2). Thus, we will assume that the function $\varphi_1(\xi)$ has a singularity at the corner point of the joint of the regions $A(\delta, \eta)$

$$\varphi_1(\xi) = P_1(\eta - \xi)^{\alpha - 1}, \quad \xi \to \eta \tag{3.5}$$

while the functions $f_i(\xi)$ are continuous in the regions in which they are defined, but their derivatives also suffer discontinuities at the corner points.

In the neighbourhood of the point $A(\delta, \eta)$

$$\begin{aligned}&f_1'(\xi) = Q_1(\eta - \xi)^{\alpha - 1} \text{ as } \xi \to \eta; \quad f_2'(\xi) = Q_2(\delta - \xi)^{\alpha - 1} \text{ as } \xi \to \delta\\&f_4'(\xi) = \bar{Q}_4 \xi^{\alpha - 1} \text{ as } \xi \to 0\end{aligned} \tag{3.6}$$

In the neighbourhood of the corner point $B(\delta_2, \eta)$ of the region $\bar{G}^{(2)}$

$$f_3'(\xi) = Q_3(\eta - \xi)^{\gamma - 1} \text{ as } \xi \to \eta; \quad f_4'(\xi) = Q_4(\delta_2 - \xi)^{\gamma - 1} \text{ as } \xi \to \delta_2 \tag{3.7}$$

We will denote by $\alpha$ and $\gamma$ the parameters characterizing the similarities of the required functions at these points, and by $P_1, Q_1, \ldots, Q_4$ arbitrary constants.

Carrying out the integration in formulae (3.5)–(3.7), we can determine the asymptotic form of the Fourier coefficients of the functions considered in the neighbourhoods of the points $A$ and $B$. Taking into account that there are no singularities at the corner points on the right-hand sides of the system of integral equations (3.2), using the usual method [1, 2] we can reduce it to the form

$$\begin{gathered}
-m_{12}s_\alpha H_1+2(2-m_{12})s_\alpha R_1+2n^{(1)}\alpha R_2+2n^{(2)}\alpha\bar{R}_4 = 0\\
(m_{12}+2)s_\alpha H_1+2m_{12}s_\alpha R_1-2(1-n^{(1)}\alpha)R_2-2(1-n^{(2)}\alpha)\bar{R}_4 = 0\\
\left(\frac{1}{n^{(1)}}+\alpha\right)H_1+2\alpha R_1+2s_\alpha R_2 = 0,\quad \left(\frac{1}{n^{(2)}}+\alpha\right)H_1+2\alpha R_1+2s_\alpha\bar{R}_4 = 0\\
s_\gamma R_3+\gamma R_4 = 0,\quad \gamma R_3+s_\gamma R_4 = 0
\end{gathered} \tag{3.8}$$

Here

$$H_1 = -2P_1\Gamma(\alpha)s_\alpha,\quad R_i = 2Q_i\Gamma(\alpha)s_\alpha,\quad \bar{R}_4 = 2\bar{Q}_4\Gamma(\alpha)s_\alpha,\quad R_l = 2Q_l\Gamma(\gamma)s_\gamma$$

$$s_\alpha = \sin\frac{\pi\alpha}{2},\quad s_\gamma = \sin\frac{\pi\gamma}{2},\quad m_{12} = \frac{1}{N_{11}^{(1)}}+\frac{1}{N_{11}^{(2)}},\quad n^{(m)} = 1-\frac{1}{N_{11}^{(m)}}$$

$\Gamma(\alpha)$ is the gamma function, $i = 1, 2$ and $l = 3, 4$.

A special feature of system (3.8) is the fact that it can be split into two parts: the first four equations contain the unknowns $H_1$, $R_1$, $R_2$, $\bar{R}_4$ and define the value of the parameter $\alpha$ – a feature of the characteristics of the wave field at the point $A$. The fifth and sixth equations define a singularity at the external corner point $B$ and contain the constants $R_3$ and $R_4$. Obviously the latter will not be equal to zero if the parameter $\gamma$ satisfies the equation

$$\sin^2\frac{\pi\gamma}{2}-\gamma^2 = 0 \tag{3.9}$$

For different boundary conditions we investigated [8] how the order of the singularity of the field of static stresses at the vertex of a single wedge depends on its aperture angle (Eq. (3.9) corresponds to Eq. (15) of [8] for a single wedge with unsecured surfaces and an aperture angle of 90°). As can be seen, the form of the singularity of the mechanical field at the point $B$ is independent of the constants of elasticity of the regions $\bar{G}^{(m)}$. Taking into account the mechanical meaning of the functions $f_3(y)$ and $f_4(x)$ and requiring that the energy of the whole system should be limited, we arrive at the conclusion that in Eq. (3.9) we must take into account only the real root $\gamma_0 = 1$ and a denumerable set of complex roots [2, 9] $\gamma_k$. Naturally the conditions $\mathrm{Re}\,\gamma_k > 0$ must be satisfied.

We will now consider the singularities that occur at the internal corner point of the region and which affect the stress concentration in its neighbourhood on the interface of the media. The parameter $\alpha$, which represents the singularity at the internal corner point of the region, is bound from the condition for a non-trivial solution of the first four equations of system (3.8) to exist. This system is symmetrical with respect to the elasticity parameters of the regions $G^{(m)}$ and will not change when $N_{11}^{(1)}$ is replaced by $N_{11}^{(2)}$ and vice versa. This can easily be proved if, after making this replacement in the determinant of this system, the third and fourth rows are interchanged, and then the third and fourth rows.

We will first investigate the extremely interesting question of how the parameter $\alpha$ depends on the constants of elasticity of the joined regions. Such an investigation has already been carried out in the static case [10] and it was shown that the stress field should depend on the constants of elasticity in terms of only two parameters, introduced by Dundurs

$$\alpha^* = \frac{\alpha_-}{\alpha_+},\quad \beta^* = \frac{\alpha_- -(\mu^{(2)}-\mu^{(1)})/2}{\alpha_+};\quad \alpha_\pm = \mu^{(2)}(1-\nu^{(1)})\pm\mu^{(1)}(1-\nu^{(2)})$$

If the Dundurs parameter $D^* = \alpha^*(\alpha^*-2\beta^*)$ is non-negative, at the corner points there will be a local singularity in stress values. If $D^* < 0$, there will be no singularities. When carrying out the analysis the values of the Dundurs parameter were calculated for the majority of sets of known materials, and material-stress concentrator pairs were obtained.

Table 1

| Group | Materials | $D^*$ | $G_*$ | $\alpha$ |
|---|---|---|---|---|
| 1 | Fused quartz-magnesium | 0.046 | 1.172 | 0.971 |
| | Fused quartz-tin | 0.022 | 0.551 | 0.984 |
| | Tungsten–aluminium | 0.267 | 5.642 | 0.820 |
| | Glass–lead | 0.279 | 0.603 | 0.865 |
| | Brass–molybdenum | 0.111 | 0.612 | 0.948 |
| | Nickel–gold | 0.159 | 0.808 | 0.934 |
| | Titanium–magnesium | 0.122 | 2.399 | 0.941 |
| 2 | Aluminium–steel | 0.140 | 0.374 | 0.937 |
| | Magnesium–copper | 0.114 | 0.276 | 0.942 |
| | Magnesium–zinc | 0.083 | 0.388 | 0.969 |
| | Magnesium–platinum | 0.173 | 0.132 | 0.921 |
| 3 | Zinc–platinum | - 0.04 | 0.391 | 1.000 |
| | Steel–steel | 0 | 1 | 1 |
| | Aluminium–aluminium | 0 | 1 | 1 |

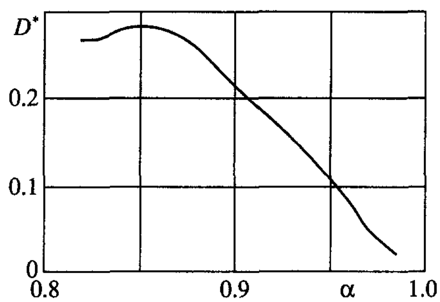


Fig. 1

We will investigate the relation $\alpha = \alpha(D^*)$ for three combinations of materials of the joined regions. The first group of pairs of materials will include material-concentrators, which are characterized by the occurrence of local singularities in the stress values at the corner points. The second group includes materials, at the interface of which Stoneley waves occur [3]. Finally, the third group includes pairs of materials which do not occur in the first two groups, including pairs consisting of similar materials (these pairs are usually called "ordinary").

In Table 1, for all three groups of pairs of materials, we show values of the least positive roots $\alpha$, obtained from the condition for a non-trivial solution of the first four equations of system (3.8) to exist. The first position in the pair is occupied by the material of the inner region, while the second position is occupied by the fused material. For each pair we calculated the parameter $G$, which defines the ratio of the wave impedances of the materials [7] of the regions $G^{(1)}$ and $G^{(2)}$, and also the Dundurs parameter $D^*$, which enables us to judge the value of the local singularity of the stresses at the joined point $A$.

Numerical analysis of the determinant of the system and the data in Table 1 showed that, for material-concentrators, there are only real roots $0 < \alpha < 1$. It should be noted that the dependence of the value of the singularity parameter $\alpha$ on the Dundurs parameter $D^*$ and on the ratio of the constants of elasticity of the materials of the regions $G^{(m)}$ is fairly complex. However, by a numerical analysis we were able to establish that: (1) for fixed elastic parameters of one of the regions, a reduction in the stiffness of the other leads to an increase in the value of $\alpha$, and (2) the relation $\alpha = \alpha(D^*)$ has a more complex form: for the same value of the Dundurs parameter $D^*$ different values of the root investigated $0 < \alpha < 1$ can exist; this is illustrated in Fig 1, where we show this relationship.

## 4. NUMERICAL ANALYSIS

Using Bubnov's method to solve system (3.2) and expanding the hyperbolic and trigonometric functions which occur in the expressions for the operators $M_{k1}$ and $L_{kr}$ from this system in terms of the functions

Table 2

| $\Omega^{(1)}$ | $E$ | $x/\delta$ | | | | $\hat{x}/\delta_2$ | |
|---|---|---|---|---|---|---|---|
| | | 0.0–0.2 | 0.2–0.4 | 0.6–0.8 | 0.8–1.0 | 0.0–0.2 | 0.8–1.0 |
| Titanium–magnesium | | | | | | | |
| 0.192 | 0.084 | 9.1 | 10.4 | 8.8 | 12.1 | 10.6 | 9.5 |
| 0.269 | 1.112 | 10.1 | 8.0 | 7.3 | 12.3 | 11.9 | 9.0 |
| 0.543 | 26.230 | 9.6 | 7.4 | 7.8 | 16.7 | 17.3 | 31.2 |
| Copper–silver | | | | | | | |
| 0.185 | 0.078 | 7.3 | 8.1 | 6.6 | 10.3 | 21.8 | 9.3 |
| 0.241 | 1.154 | 8.4 | 6.6 | 5.1 | 9.1 | 20.6 | 9.1 |
| 0.645 | 6.238 | 7.2 | 6.8 | 6.3 | 11.1 | 21.3 | 14.6 |

$$\cos\alpha_k(y-\eta),\quad \sin\alpha_k(y-\eta),\quad \cos\beta_j(x-\delta),\quad \cos\chi_j(\hat{x}-\delta_2)$$

we obtain from boundary conditions (3.1) an infinite system of algebraic equations for determining the Fourier coefficients $f_{10}, f_{1k}, \ldots, \varphi_{4j}, \varphi_{1k}$. Such an asymptotic analysis enables us to reduce this system to a finite system, since for large $j$ and $k$ we can replace the Fourier coefficients by their asymptotic forms, defined by the values of the parameters of the singularities of the wave field at the points $A$ and $B$. For an integrated description of the features of the edge mode and the whole spectrum of natural frequencies, it is best to make a comparative analysis of the energy characteristic – the mean energy $E$ stored over a period in the elastic region (or its element) [1–4]. The part of the region of the section situated in the first quarter, which is of interest, is split into vertical parts and for each of them we calculate the value of the stored energy. The ratio of this energy to the total energy, calculated for each part of the splitting and expressed as a percentage, represents the energy distribution over the cross-section of the prism.

The main conclusion of our numerical analysis of the spectrum of the natural frequencies is the fact that the oscillations of the region, made up of material-concentrators, for small values of $\delta_2$ are accompanied by a more pronounced "edge" nature compared with the materials of the second group. This can obviously be explained by the closeness of the interface of the media, where a stress concentration occurs. When the parameter $\delta_2$ increases the effects characteristic for the phenomena of edge resonance [3] are much less appreciable. At the same time, an increase in the parameter $\delta_2$ to approximately the value of $\delta$, only slightly increases the "boundary" resonance effects [5] at the interface for material-concentrators. This is illustrated by the data presented in the upper part of Table 2, for a region consisting of titanium–magnesium materials with values of the parameters $L = 1$, $\delta = 0.5$ and $\delta_2 = 0.5$. The third row of the table corresponds to the edge resonance frequency. In the lower part of Table 2 we present data for the region consisting of a copper-silver pair of materials, which are not concentrators, with the same geometrical parameters.

When the dimensions of the external regions are increased further, the characteristics of the edge resonance disappear in practice for any combinations of materials.

When investigating the characteristics of the wave field in the neighbourhood of the interface of the regions, the error in satisfying the matching conditions with respect to the displacements did not exceed 2%, and with respect to the stresses – 5–6%. When there is a singularity at the point $A$, the series in the stresses diverge when $x = \delta$, and to satisfy the matching conditions over the whole section $|y| \leq \eta$, separation of the singularity and the use of generalized methods of summation and regularization are necessary [11].

We will consider the distribution of the normalized components of the stress tensor, i.e. referred to the maximum values, for different combinations of pairs of material-concentrators for the first, second and third natural frequencies. It should be noted that in a numerical analysis it is important to take into account not only the elastic parameters of the pairs of materials, but also the acoustic impedances of the contacting media. As was noted previously in [7], when analysing the features of the wave field in an infinite composite waveguide in the case of different ratios of the acoustic impedances of the materials of the regions $G^{(m)}$, a different degree of effectiveness of the reflection of the wave field from the interface is observed. For combinations of pairs of materials corresponding to the case $G_* > 1$, localization of the stresses in the inner corner points of the region is observed.

In Fig. 2(a) we show the distribution of the normalized stress $\bar{\sigma}_{11}$ $(x, 0.95)$ for a pair of zinc-gold concentrators with $L = a/b = 2.25$, which corresponds to the middle plateau in the spectrum of natural

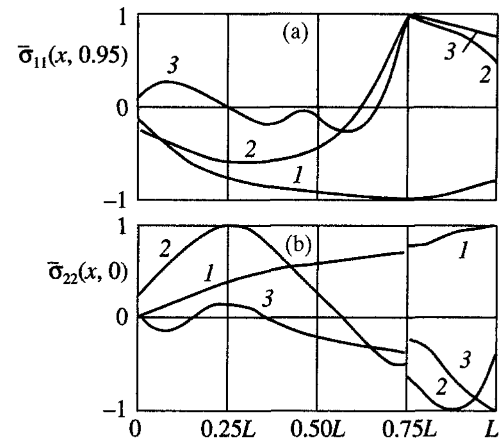


Fig. 2

frequencies [3]. The numbers of the natural frequencies are indicated on the curves. The third natural frequency is the frequency of the boundary resonance: in a form corresponding to it, the zone of high stresses is concentrated in the inner corner point of the region and in the region of the ends of the rectangle. The stress values fall sharply as one moves away from the interface.

In Fig. 2(b) we show the distribution of the normalized stress $\bar{\sigma}_{22}(x, 0)$ along the $y = 0$ axis for the same pair of materials and the same geometrical dimensions of the region. In this case, we again observe a zone of relatively high stresses at the internal corner points of the region, but there is no localization along the boundary. Moreover, at the interface of the regions the natural forms undergo a sudden discontinuity, which, in principle, agrees with Plevako's conclusions [12].

For a combination of pairs of material-concentrators, corresponding to $G_* < 1$ (the inner region is stiffer), we have a qualitatively different pattern in the stress distribution. Obviously the energy distribution depends not only on the combination of materials, which make up the region, but also on the value of the frequency parameter $\Omega^{(1)} = \omega a/c_S^{(1)}$ ($c_S^{(1)}$ is the velocity of shear waves in the region $G^{(1)}$). In the case considered, the pair of material-concentrators at low frequencies has an almost uniform energy distribution over the cross-section area. At the boundary-resonance frequency, however, a large part of the stored energy is concentrated in the external less rigid region $G^{(2)}$. No stress concentration is observed either at the interface of the regions or in the region of the external corner points of the composite section considered. However, about two-thirds of the stored energy is concentrated in the external region $G^{(2)}$, which is less stiff than the internal region. Hence, we cannot say that it is only the Dundurs parameters that determine the material-concentrator pairs. At an internal corner point of the region, a stress concentration occurs for a certain combination of these materials. In this case, the external region must be stiffer than the inner region.

The question of the features of the behaviour of the dynamic characteristics for pairs of materials for which the occurrence of Stoneley waves is possible is also of interest. And in this case, the composition of the materials is also important. The results of an analysis for pairs of materials of the second group has shown that the occurrence at the interface of two media of Stoneley type waves is only possible when, for the materials making up the region, the stiffness $G_* < 1$. In this case one observes a pronounced localization of the stresses along the interface. At an internal corner point of the regions there is also a singularity, which is confirmed by a numerical analysis of the characteristic equation of system (3.8) for pairs of materials with given properties. Moreover, we have an abrupt discontinuity in the stresses $\bar{\sigma}_{22}(\delta, y)$ at the interface. As regards the quantity of stored energy, about three-quarters of it is concentrated in the external, "stiffer", region. This confirms the previous conclusions that, for such media, boundary resonance characteristics are present [13], which, however, are not as pronounced as in the case of a uniform region. The internal, "softer", region absorbs about one-quarter of the stored energy.

An investigation of the distribution of the components of the stress tensor in the neighbourhood of the boundary $x = \delta$ with stiffness parameter $G_* > 1$ has shown that, for such materials, there is no localization of the stresses along the interface of the media and at the corner point $A$. For such a combination of materials, about three-quarters of the stored energy is concentrated in the internal, "stiffer", region.

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